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Exponential representation of Jordanian matrix quantum group $GL_h(2)$

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Abstract. An exponential representation of the Jordanian matrix quantum group $GL_h(2)$ is constructed in explicit form by the generators of the classical Lie algebra $sl^*(2, C)$.

1. Introduction

Realizations of generators of quantum algebras and groups [1, 2] by the base elements of related undeformed systems can be useful for development of general theory as well as for the solution of concrete problems. It seems that a realization of this sort was first given by Jimbo in his well known paper [3] for the Borel subalgebras of the standard quantized simple Lie algebras. In [4] a deforming map (in the author's terminology) was constructed, which represents the generators of the quantum algebra $su_q(2)$ by the undeformed ones. A similar map was recently obtained for the Jordanian deformation of $sl(2)$ [5]. In [6] a method was suggested which is suitable for any standard quantized group related to the classical Cartan list. This method is based on the quantum algebra homomorphism constructed by Jimbo in [3] and the specific properties of the Gauss decomposition of these groups. The quantum variant of the decomposition of number matrices was discussed in many papers from different points of view (see, for example, [7], the brief review [8] and references therein). Let us recall that the Gauss decomposition of a matrix quantum group is defined as a transition from the $n \times n$ matrix $\boldsymbol{T} = (t_{ij})$, $i, j = 1, 2, \ldots, n$ of the original generators satisfying the matrix RTT-relation [2] (with a square number matrix R of order n^2)

$$R\boldsymbol{T}_1\boldsymbol{T}_2 = \boldsymbol{T}_2\boldsymbol{T}_1R \tag{1}$$

to the new matrices of Gauss generators $(\boldsymbol{T}_L, \boldsymbol{T}_D, \boldsymbol{T}_U)$ connected with $\boldsymbol{T}$ by the usual classical formula $\boldsymbol{T} = \boldsymbol{T}_L\boldsymbol{T}_D\boldsymbol{T}_U$. Here $\boldsymbol{T}_L$ and $\boldsymbol{T}_U$ are, respectively, strictly lower- and upper- triangular matrices with units at their diagonals, $\boldsymbol{T}_D$ is a diagonal matrix. In the relation (1) standard notation $\boldsymbol{T}_1 = \boldsymbol{T} \otimes \mathbb{1}$, $\boldsymbol{T}_2 = \mathbb{1} \otimes \boldsymbol{T}$ is used, where $\mathbb{1}$ is a unit matrix.

There are two constructive procedures to obtain the Gauss decomposition for any matrix quantum group $\boldsymbol{T}$: non-commutative generalization of the usual Gauss algorithm and contraction [8]. Both procedures have restrictions but they lead to the same result in the case of the standard quantized groups of the classical series. The main attractive properties of Gauss generators of these groups, which permit us to use the Jimbo homomorphism, are still quadratic commutation relations among them and especially mutual commutativity of every element of $\boldsymbol{T}_L$ with every element of $\boldsymbol{T}_U$. This is the reason why we can consider the problem of realization of quantum generators by the classical ones (or by operators of creation and annihilation [9]) separately for each Borel subgroup of a quantum group. These subgroups relate to the matrices $\boldsymbol{T}^{(-)} = \boldsymbol{T}_L\boldsymbol{T}_D$ and $\boldsymbol{T}^{(+)} = \boldsymbol{T}_D\boldsymbol{T}_U$. It can be shown [6] with the Faddeev, Reshetikhin and Takhtajan (FRT) framework that $\boldsymbol{T}^{(\pm)}$ are isomorphic to the Borel subalgebras $\boldsymbol{L}^{(\pm)}$ of the corresponding dual quantum algebras for which the Jimbo homomorphism exists. In this way, for example, we obtain for the quantum group $SL_q(2)$ the following representation [6]:

$$\boldsymbol{T} = \begin{pmatrix} q^{-H/2} \otimes q^{H/2} & fq^{-H/2} \otimes X^{(+)} \\ gX^{(-)} \otimes q^{H/2} & q^{H/2} \otimes q^{-H/2} + fgX^{(-)} \otimes X^{(+)} \end{pmatrix} \tag{2}$$

where $X^{(\pm)}$, H are the generators of the classical algebra $sl(2, C)$ with the usual commutation relations $[H, X^{(\pm)}] = \pm 2X^{(\pm)}$, $[X^{(+)}, X^{(-)}] = H$ and f, g are arbitrary constants.

As quantum groups are dual structures of the relate quantum algebras it is natural to expect that the above representation (2) can be simplified if one uses the generators of the dual Lie algebra $sl_s^*(2, C)$ instead of $sl(2, C)$. Recall [1] that commutation relations among generators of the Lie algebra, which are dual to any quasitriangular Lie bialgebra (g, δ_s), are defined by the 1-cocycle δ_s associated with a classical r-matrix. In the case of $SL_q(2)$ such an r-matrix can be obtained from the matrix

$$R_s(q) = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & q - q^{-1} & 1 & 0 \\ 0 & 0 & 0 & q \end{pmatrix} \tag{3}$$

by $r_s = \frac{\mathrm{d}R_s}{\mathrm{d}q}\big|_{q=1}$ and it yields for the $sl_s^*(2, C)$-generators $\widetilde{H}_s$, $\widetilde{X}_s^{(\pm)}$ the following commutation relations:

$$[\widetilde{H}_s, \widetilde{X}_s^{(\pm)}] = \widetilde{X}_s^{(\pm)} \qquad [\widetilde{X}_s^{(+)}, \widetilde{X}_s^{(-)}] = 0. \tag{4}$$

Using (4), we obtain the realization

$$\boldsymbol{T} = \begin{pmatrix} q^{\widetilde{H}_s} & fq^{\widetilde{H}_s}\widetilde{X}_s^{(+)} \\ g\widetilde{X}_s^{(-)}q^{\widetilde{H}_s} & q^{-\widetilde{H}_s} + fg\widetilde{X}_s^{(-)}q^{\widetilde{H}_s}\widetilde{X}_s^{(+)} \end{pmatrix}. \tag{5}$$

Representations of the form (2) or (5) can be constructed for a quantum group provided that this group has the Gauss decomposition with the properties mentioned above. However, there are quantum groups without such properties. For example, in the case of the Jordanian quantum group $GL_h(2)$ [10, 11] the contraction procedure gives only partial decomposition, whereas commutation relations among new generators, obtained by the Gauss algorithm, are non-quadratic ones. For this reason in the present work we shall consider the representation of the Jordanian quantum group which is similar to the classical exponential representation of elements of Lie groups. The representation of this form was studied in [12] for the group $GL_q(2)$. The authors of [12] have taken into account the following exponential-like property of the $GL_q(2)$ $\boldsymbol{T}$-matrix:

$$R_s(q^2)\boldsymbol{T}_1'\boldsymbol{T}_2' = \boldsymbol{T}_2'\boldsymbol{T}_1'R_s(q^2) \tag{6}$$

where $\boldsymbol{T}' = \boldsymbol{T}^2$ and $R_s(q^2)$ is the R-matrix (3) with q replaced by q^2 (note that $R_s(q^2) \neq R_s(q)^2$). The property (6) allowed them to assume the exponential structure of the $GL_q(2)$ $\boldsymbol{T}$-matrix

$$\boldsymbol{T} = \mathrm{e}^{hM} = \sum_{k=0} \frac{(h\boldsymbol{M})^k}{k!} \qquad \mathrm{e}^h = q. \tag{7}$$

The elements of the matrix $\boldsymbol{M}$ are exactly the mentioned $sl_s^*(2, C)$-generators $\widetilde{H}_s$, $\widetilde{X}_s^{(\pm)}$ completed by the central elements $\widetilde{C}_s$

$$\boldsymbol{M} = \begin{pmatrix} \widetilde{H}_s + \widetilde{C}_s & \widetilde{X}_s^{(+)} \\ \widetilde{X}_s^{(-)} & -\widetilde{H}_s + \widetilde{C}_s \end{pmatrix}.$$

In the paper [12] it was proven that the series in h defined by (7) with coefficients from the universal enveloping algebra $\mathcal{U}(sl_s^*(2, C))$ produce the generators of $GL_q(2)$, however, the formulae were not given in closed form. Here we shall consider a similar representation for the Jordanian quantum group $GL_h(2)$ and obtain the explicit expressions for its generators.

2. Exponential representation

The Jordanian R-matrix is the one-parameter number matrix

$$R(h) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -h & 1 & 0 & 0 \\ h & 0 & 1 & 0 \\ h^2 & -h & h & 1 \end{pmatrix} \tag{8}$$

where h is a deformation parameter. This matrix satisfies the remarkable functional equation

$$R(h_1)R(h_2) = R(h_1 + h_2). \tag{9}$$

Equation (9) implies that the quantum R-matrix (8) can be written as an exponential function $R = \exp(hr)$ of the number classical r-matrix

$$r = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \end{pmatrix}. \tag{10}$$

Using the RTT-equation (1) with R given by (8), we obtain for the $GL_h(2)$ generators $\boldsymbol{T} = (t_{ij})$, $i, j = 1, 2$ the following quadratic relations [11] ([10] for the $h = 1$ case):

$$[t_{11}, t_{12}] = -ht_{12}^2 \tag{11}$$

$$[t_{11}, t_{21}] = h(t_{11}^2 + t_{12}t_{21} - t_{11}t_{22} - ht_{12}t_{22}) = h(t_{11}^2 + t_{21}t_{12} - t_{22}t_{11} + ht_{22}t_{12}) \tag{12}$$

$$[t_{11}, t_{22}] = h(t_{11}t_{12} - t_{22}t_{12}) = h(t_{12}t_{11} - t_{12}t_{22}) \tag{13}$$

$$[t_{12}, t_{21}] = h(t_{11}t_{12} + t_{12}t_{22}) = h(t_{12}t_{11} + t_{22}t_{12}) \tag{14}$$

$$[t_{12}, t_{22}] = ht_{12}^2 \tag{15}$$

$$[t_{21}, t_{22}] = -h(t_{22}^2 - t_{11}t_{22} + t_{21}t_{12} + ht_{11}t_{12}) = -h(t_{22}^2 + t_{12}t_{21} - t_{22}t_{11} - ht_{12}t_{11}). \tag{16}$$

The quantum determinant (central element) has the form

$$\det{}_h \boldsymbol{T} = \mathcal{D} = t_{11}t_{22} - t_{12}t_{21} + ht_{12}t_{22}. \tag{17}$$

Taking into account expression (17), we can rewrite the formulae (12) and (16) in the shortened form

$$[t_{11}, t_{21}] = h(t_{11}^2 - \mathcal{D}) \qquad [t_{21}, t_{22}] = h(\mathcal{D} - t_{22}^2).$$

It is not difficult to show by direct calculation that the quadratic relations (11)–(16) imply equation (6) with the Jordanian R-matrix (8) of doubled argument $R(2h)$

$$R(2h)\boldsymbol{T}_1'\boldsymbol{T}_2' = \boldsymbol{T}_2'\boldsymbol{T}_1'R(2h).$$

As the previous equation holds for arbitrary values of the parameter h, it is naturally to assume, following [12], that the $\boldsymbol{T}$-matrix of the Jordanian quantum group can be expressed as an exponential of the form (7)

$$\boldsymbol{T} = \mathrm{e}^{h\boldsymbol{M}}.$$

Expand the R-matrix (8) in a series in the parameter h and substitute it together with the series (7) in the RTT-relation. The matrix coefficients at the second degree of h give us the equation

$$[\boldsymbol{M}_1, \boldsymbol{M}_2] = [\boldsymbol{M}_1 + \boldsymbol{M}_2, r] \tag{18}$$

which is the representation of the well known Poisson brackets in the quantum inverse scattering method [13, 14]. Equation (18) yields for the matrix elements m_{ij}, $i, j = 1, 2$ the following commutation relations of a solvable Lie algebra:

$$[m_{11}, m_{12}] = [m_{11}, m_{22}] = [m_{12}, m_{22}] = 0 \qquad [m_{12}, m_{21}] = 2m_{12}$$

$$[m_{11}, m_{21}] = [m_{21}, m_{22}] = m_{11} - m_{22}.$$

Let us denote $m_{11} = \widetilde{X}^{(-)} + \widetilde{C}$, $m_{12} = \widetilde{X}^{(+)}$, $m_{21} = \widetilde{H}$, $m_{22} = -\widetilde{X}^{(-)} + \widetilde{C}$ and rewrite these formulae as

$$[\widetilde{X}^{(\pm)}, \widetilde{H}] = 2\widetilde{X}^{(\pm)} \qquad [\widetilde{X}^{(+)}, \widetilde{X}^{(-)}] = 0 \qquad [\widetilde{C}, \cdot] = 0. \tag{19}$$

Note that the generators $\widetilde{H}, \widetilde{X}^{(\pm)}, \widetilde{C}$ are linearly combined with $\widetilde{H}_s, \widetilde{X}_s^{(\pm)}, \widetilde{C}_s$ [12] (see the introduction).

Divide the matrix $\boldsymbol{M}$ into two parts $\boldsymbol{M} = \widetilde{C}\mathbb{1} + \boldsymbol{M}_s$. Since $\widetilde{C}$ is a central element, the matrix $\boldsymbol{T}$ can be factored as

$$\boldsymbol{T} = \mathrm{e}^{h\widetilde{C}\mathbb{1}}\mathrm{e}^{h\boldsymbol{M}_s}. \tag{20}$$

We shall see below that the second factor in (20) $\boldsymbol{T}_s = \mathrm{e}^{h\boldsymbol{M}_s}$ determines the exponential representation of $SL_h(2)$. To obtain such a representation consider

$$\boldsymbol{T}_s = \begin{pmatrix} t_{11}^s & t_{12}^s \\ t_{21}^s & t_{22}^s \end{pmatrix} = \sum_{n=0} \frac{(h\boldsymbol{M}_s)^n}{n!}.$$

Put

$$\boldsymbol{M}_s^n = \begin{pmatrix} m_{11}^{(n)} & m_{12}^{(n)} \\ m_{21}^{(n)} & m_{22}^{(n)} \end{pmatrix} \qquad \boldsymbol{M}_s^0 = \mathbb{1} \qquad \boldsymbol{M}_s^1 = \boldsymbol{M}_s = \begin{pmatrix} \widetilde{X}^{(-)} & \widetilde{X}^{(+)} \\ \widetilde{H} & -\widetilde{X}^{(-)} \end{pmatrix}.$$

From $\boldsymbol{M}_s^{n+1} = \boldsymbol{M}_s\boldsymbol{M}_s^n$ for the elements $m_{ij}^{(n)}$ one can easily write out the following recursion relations:

$$m_{11}^{(n+1)} = \widetilde{X}^{(-)}m_{11}^{(n)} + \widetilde{X}^{(+)}m_{21}^{(n)} \qquad m_{21}^{(n+1)} = \widetilde{H}m_{11}^{(n)} - \widetilde{X}^{(-)}m_{21}^{(n)} \tag{21}$$

$$m_{12}^{(n+1)} = \widetilde{X}^{(-)}m_{12}^{(n)} + \widetilde{X}^{(+)}m_{22}^{(n)} \qquad m_{22}^{(n+1)} = \widetilde{H}m_{12}^{(n)} - \widetilde{X}^{(-)}m_{22}^{(n)}. \tag{22}$$

The relations (21) and (22) are organized as two independent but identical in form pairs. Consider, for example, the first of them. We obtain by the iteration

$$m_{11}^{(n+1)} = ((\widetilde{X}^{(-)})^2 + \widetilde{X}^{(+)}\widetilde{H})m_{11}^{(n-1)}. \tag{23}$$

It is convenient to denote the determinant-like expression appearing in (23) by $\boldsymbol{d}^2 = (\widetilde{X}^{(-)})^2 + \widetilde{X}^{(+)}\widetilde{H}$. Introduce, as well, $\boldsymbol{t}^2 = (\widetilde{X}^{(-)})^2 + \widetilde{H}\widetilde{X}^{(+)}$ which connects with $\boldsymbol{d}^2$ by the formula $\boldsymbol{t}^2 = \boldsymbol{d}^2 - 2\widetilde{X}^{(+)}$. Note that both $\boldsymbol{d}^2$ and $\boldsymbol{t}^2$ are not central elements for the Lie algebra generated by $\widetilde{X}^{(+)}$, $\widetilde{X}^{(-)}$ and $\widetilde{H}$. From the recursion relation (23), taking into account $m_{11}^{(0)} = 1$, $m_{11}^{(1)} = \widetilde{X}^{(-)}$, we obtain the solution containing only even degrees of $\boldsymbol{d}$:

$$m_{11}^{(2n)} = \boldsymbol{d}^{2n} \qquad m_{11}^{(2n+1)} = \boldsymbol{d}^{2n}\widetilde{X}^{(-)}. \tag{24}$$

Using this solution we obtain formally

$$t_{11}^s = \sum_{n=0} \frac{h^{2n}}{2n!}\boldsymbol{d}^{2n} + \sum_{n=0} \frac{h^{2n+1}}{(2n+1)!}\boldsymbol{d}^{2n}\widetilde{X}^{(-)} = \cosh(h\boldsymbol{d}) + \boldsymbol{d}^{-1}\sinh(h\boldsymbol{d})\widetilde{X}^{(-)}.$$

The relation (A2) from the appendix permits us to rewrite the expression for t_{11}^s in terms of $\boldsymbol{t}$ instead of $\boldsymbol{d}$,

$$t_{11}^s = \cosh(h\boldsymbol{t}) + (h\widetilde{X}^{(+)} + \widetilde{X}^{(-)})\boldsymbol{t}^{-1}\sinh(h\boldsymbol{t}). \tag{25}$$

Note that the above series give no a representation of t_{11}^s in the standard basis of a universal enveloping algebra.

Now return to recursion relations (21) and consider the second of them

$$\widetilde{X}^{(+)}m_{21}^{(n)} = m_{11}^{(n+1)} - \widetilde{X}^{(-)}m_{11}^{(n)}.$$

Substituting the elements (24) in this relation we obtain for the even coefficients

$$\widetilde{X}^{(+)}m_{21}^{(2n)} = \boldsymbol{d}^{2n}\widetilde{X}^{(-)}(\boldsymbol{t}^{2n} - \boldsymbol{d}^{2n}) = \widetilde{X}^{(+)}(-2n\widetilde{X}^{(-)}\boldsymbol{t}^{2(n-1)}).$$

Since similar calculations can be carried out for the odd coefficients, we obtain

$$m_{21}^{(2n)} = -2n\widetilde{X}^{(-)}\boldsymbol{t}^{2(n-1)} \qquad m_{21}^{(2n+1)} = (2n + \widetilde{H})\boldsymbol{t}^{2n}.$$

The summation leads to the expression

$$t_{21}^s = h\cosh(h\boldsymbol{t}) + (\widetilde{H} - h\widetilde{X}^{(-)} - 1)\boldsymbol{t}^{-1}\sinh(h\boldsymbol{t}). \tag{26}$$

The other pair of recursion relations (22) can be solved by the same manner and the resulting formulae have the form

$$t_{12}^s = \widetilde{X}^{(+)}\boldsymbol{t}^{-1}\sinh(h\boldsymbol{t}) \qquad t_{22}^s = \cosh(h\boldsymbol{t}) - \widetilde{X}^{(-)}\boldsymbol{t}^{-1}\sinh(h\boldsymbol{t}). \tag{27}$$

As the final step it should be checked that these expressions satisfy the quantum group commutation relations (11)–(16). Let us note, to simplify the check slightly, that the Jordanian R-matrix (8) commutes with the number matrix $A_\otimes(h)$:

$$A_\otimes(h) = A(h)\otimes A(h) = \begin{pmatrix} 1 & 0 \\ -h & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ -h & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -h & 1 & 0 & 0 \\ -h & 0 & 1 & 0 \\ h^2 & -h & -h & 1 \end{pmatrix}. \tag{28}$$

Therefore, the elements of the matrix

$$\boldsymbol{T}_s' = \boldsymbol{T}A(h) = \begin{pmatrix} t_{11}^s - ht_{12}^s & t_{12}^s \\ t_{21}^s - ht_{22}^s & t_{22}^s \end{pmatrix}$$

satisfy the RTT-equation (1) if and only if the elements of $\boldsymbol{T}_s$ satisfy it. Using the auxiliary commutators given in the appendix by the formulae (A3), we can easily check the group relations (11)–(16) and the identity

$$\det{}_h \boldsymbol{T}_s = 1.$$

Hence, the expressions (25)–(27) give us in closed form the realization of the $SL_h(2)$ generators. Multiplying each of these formulae by $\mathrm{e}^{h\widetilde{C}}$ (see (20)), we obtain the desired exponential representation of the Jordanian quantum group $GL_h(2)$.

Leaving out possible applications of the obtained formulae for construction of $GL_h(2)$ representations, let us note that the exponential form (7) implies the formula

$$\boldsymbol{T}(h_1)\boldsymbol{T}(h_2) = \boldsymbol{T}(h_1 + h_2). \tag{29}$$

Considering $\boldsymbol{T}(h)$ as the one-parameter set of matrices with non-commutative entries we can define their action on a quantum vector $\boldsymbol{X}$ by

$$\boldsymbol{T}(h_1)\boldsymbol{X}(h_2) = \begin{pmatrix} t_{11}(h_1) & t_{12}(h_1) \\ t_{21}(h_1) & t_{22}(h_1) \end{pmatrix} \begin{pmatrix} x_1(h_2) \\ x_2(h_2) \end{pmatrix} = \begin{pmatrix} x_1(h_1 + h_2) \\ x_2(h_1 + h_2) \end{pmatrix} = \boldsymbol{X}(h_1 + h_2) \tag{30}$$

where

$$\boldsymbol{X}(h) = \boldsymbol{T}(h)\begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \text{or} \qquad \boldsymbol{X}(h) = \boldsymbol{T}(h)\begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Since $\boldsymbol{T}$ is invertible one can say that equation (30) gives the isomorphism between quantum vectors with different h.

3. Hopf structure

A Hopf algebra structure on a matrix quantum group into the FRT framework [2] is defined as

$$\Delta(\boldsymbol{T}) = \boldsymbol{T}(\otimes)\boldsymbol{T} \qquad S(\boldsymbol{T}) = \boldsymbol{T}^{-1} \qquad \epsilon(\boldsymbol{T}) = \mathbb{1}. \tag{31}$$

Here $(\otimes)$ denotes the usual matrix product of $\boldsymbol{T}$-matrices with a tensor product of their elements. The above exponential realization of the Jordanian quantum group permits us to obtain easily an antipode of each of its generator because of the calculation of the inverse matrix elements reduces to the change $h \to -h$ in the exponential or, which is the same, in the formulae (25)–(27). Using thus derived formulae we can obtain (in the case of $SL_h(2)$) the representation

$$S(\boldsymbol{T}_s) = \boldsymbol{T}_s^{-1} = \begin{pmatrix} t_{22}^s + ht_{12}^s & -t_{12}^s \\ -t_{21}^s - ht_{11}^s + ht_{22}^s + h^2 t_{12}^s & t_{11}^s - ht_{12}^s \end{pmatrix}. \tag{32}$$

Note that it is not possible to consider the transition $t_{ij}^s \to S(t_{ij}^s)$ by (32) as an involution because of $S^2(t_{ij}^s) \neq t_{ij}^s$. However, the following relation occurs:

$$S^2(\boldsymbol{T}_s) = U\boldsymbol{T}_s U^{-1} \tag{33}$$

where $U = A(2h)$ with the number matrix A defined in (28).

Using the definition of comultiplication, antipode and counity (31) one can define a heritable Hopf structure on the elements of $\boldsymbol{M}_s$ and, consequently, on the universal enveloping algebra of the solvable Lie algebra $sl^*(2, C)$. To this end we express the algebra matrix $\boldsymbol{M}_s$ via the group matrix $\boldsymbol{T}_s$

$$\boldsymbol{M}_s = h^{-1} \log \boldsymbol{T}_s. \tag{34}$$

This formula is correct since we consider the 'deformation of the unit' and $\boldsymbol{T}_s - \mathbb{1} \approx \mathrm{O}(h)$. From the definition of co-operations and equation (34) we immediately obtain

$$\begin{aligned} \Delta(\boldsymbol{M}_s) &= h^{-1} \log \boldsymbol{T}_s(\otimes)\boldsymbol{T}_s = h^{-1} \log \mathrm{e}^{\boldsymbol{M}_{s(1)}} \mathrm{e}^{\boldsymbol{M}_{s(2)}} \\ S(\boldsymbol{M}_s) &= h^{-1} \log S(\boldsymbol{T}_s) = -\boldsymbol{M}_s \\ \epsilon(\boldsymbol{M}_s) &= h^{-1} \log \epsilon(\boldsymbol{T}_s) = 0. \end{aligned} \tag{35}$$

Here $\boldsymbol{M}_{(1)} = \boldsymbol{M}(\otimes)\mathbb{1}$, $\boldsymbol{M}_{(2)} = \mathbb{1}(\otimes)\boldsymbol{M}$. According to (35), the definitions of the antipode and counity are the same as in the undeformed case, but the definition of the comultiplication is not. We derive the images of the $\boldsymbol{M}_s$-matrix elements under the comultiplication operation by the Campbell–Hausdorff formula for the first two terms

$$\Delta(\boldsymbol{M}_s) = \boldsymbol{M}_{s(1)} + \boldsymbol{M}_{s(2)} + \tfrac{1}{2} h [\boldsymbol{M}_{s(1)}, \boldsymbol{M}_{s(2)}] + \cdots.$$

For the algebra $sl^*(2, C)$ generators we obtain

$$\begin{aligned} \Delta(\widetilde{X}^{(-)}) &= \Delta_0(\widetilde{X}^{(-)}) + \tfrac{1}{2} h \widetilde{X}^{(+)} \wedge \widetilde{H} + \mathrm{O}(h^2) \\ \Delta(\widetilde{X}^{(+)}) &= \Delta_0(\widetilde{X}^{(+)}) + h \widetilde{X}^{(-)} \wedge \widetilde{X}^{(+)} + \mathrm{O}(h^2) \\ \Delta(\widetilde{H}) &= \Delta_0(\widetilde{H}) - h \widetilde{X}^{(-)} \wedge \widetilde{H} + \mathrm{O}(h^2) \end{aligned} \tag{36}$$

where $\Delta_0(A) = A \otimes 1 + 1 \otimes A$ is undeformed comultiplication. It is natural to recognize in expressions (36) the 1-cocycle δ_* of the bialgebra $(sl^*(2), \delta_*)$

$$\delta_*(\widetilde{X}^{(-)}) = \tfrac{1}{2} \widetilde{X}^{(+)} \wedge \widetilde{H} \qquad \delta_*(\widetilde{X}^{(+)}) = \widetilde{X}^{(-)} \wedge \widetilde{X}^{(+)} \qquad \delta_*(\widetilde{H}) = \widetilde{X}^{(-)} \wedge \widetilde{H}.$$

This bialgebra is dual to the triangular Lie bialgebra $(sl(2), \delta)$ which is defined by the classical r-matrix (10). Recall that the brackets and 1-cocycles of these Lie algebras are dual with respect to a non-degenerate linear form $(\cdot\,,\cdot)$

$$(\delta_*(A), X \wedge Y) = (A, [X, Y]) \qquad ([A, B]_*, X) = (A \wedge B, \delta(X))$$

where $X, Y \in sl(2)$, $A, B \in sl^*(2)$. In our case the form is defined by its values which are only distinct from zero:

$$(\widetilde{X}^{(-)}, H) = (\widetilde{H}, X^{(-)}) = (\widetilde{X}^{(+)}, X^{(+)}) = 1.$$

The general expressions for results of comultiplication (35) are cumbersome, so let us consider the simpler case of the triangular subgroup of $SL_h(2)$ which is defined by the condition $t^s_{12} = 0$. This subgroup is generated (via our representation (25)–(27)) by $\widetilde{X}^{(-)}$ and $\widetilde{H}$. Since the subalgebra generated by these generators is isomorphic to the Borel subalgebra $\boldsymbol{b}^-$ of $sl(2, C)$, we omit the tilde in what follows.

To obtain the desired formulae we put formally $\widetilde{X}^{(+)} = 0$ in the expressions (25)–(27). As a result we have

$$\begin{aligned} & t^{(-)}_{11} = \mathrm{e}^{hX^{(-)}} \qquad t^{(-)}_{22} = \mathrm{e}^{-hX^{(-)}} \\ & t^{(-)}_{21} = hH_h = (H - 1)(X^{(-)})^{-1} \sinh(hX^{(-)}) + h\mathrm{e}^{-hX^{(-)}}. \end{aligned} \tag{37}$$

Equations (37) directly imply the commutation relation

$$[H_h, X^{(-)}] = -2 \frac{\sinh(hX^{(-)})}{h} \tag{38}$$

which is the relation among the generators of Jordanian deformation of algebra $sl(2, c)$ [15]. Note here that our realization of H_h differs from those obtained in [5, 16]. To calculate $\Delta(X^{(-)})$, $\Delta(H)$ induced by group comultiplication, we use equations (31),

$$\Delta(t^{(-)}_{11}) = t^{(-)}_{11} \otimes t^{(-)}_{11} \qquad \Delta(t^{(-)}_{21}) = t^{(-)}_{21} \otimes t^{(-)}_{11} + t^{(-)}_{22} \otimes t^{(-)}_{21}.$$

From the first of those we deduce that $X^{(-)}$ is a primitive element of $\mathcal{U}_h(\boldsymbol{b}^-)$,

$$\Delta(t_{11}^{(-)}) = \mathrm{e}^{h\Delta X^{(-)}} = \mathrm{e}^{hX^{(-)}} \otimes \mathrm{e}^{hX^{(-)}} = \mathrm{e}^{h\Delta_0 X^{(-)}}.$$

Finally, using the explicit expression for $t_{21}^{(-)}$ (37) we obtain

$$\Delta(H) = 1 \otimes 1 + (H\boldsymbol{K}_0 \otimes \mathrm{e}^{hX^{(-)}} + \mathrm{e}^{-hX^{(-)}} \otimes H\boldsymbol{K}_0 \tag{39}$$

$$-\boldsymbol{K}_0 \otimes \mathrm{e}^{hX^{(-)}} - \mathrm{e}^{-hX^{(-)}} \otimes \boldsymbol{K}_0 + h\mathrm{e}^{-hX^{(-)}} \otimes \mathrm{e}^{hX^{(-)}})(\Delta\boldsymbol{K}_0)^{-1}. \tag{40}$$

In this formula

$$\boldsymbol{K}_0 = (X^{(-)})^{-1}\sinh(hX^{(-)}).$$

It can be found from $\boldsymbol{K}$ (see the appendix) by putting $\widetilde{X}^{(+)} = 0$.

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Appendix

Using the explicit expressions for $\boldsymbol{t}^2$ and $\boldsymbol{d}^2$ one can check the following quadratic relations:

$$\begin{aligned}
&[\widetilde{X}^{(\pm)}, \boldsymbol{t}^2] = 2\widetilde{X}^{(+)}\widetilde{X}^{(\pm)} \qquad && [\boldsymbol{t}^2, \widetilde{H}] = 2\boldsymbol{t}^2 + 2(\widetilde{X}^{(-)})^2 \\
&[\widetilde{X}^{(\pm)}, \boldsymbol{d}^2] = 2\widetilde{X}^{(+)}\widetilde{X}^{(\pm)} && [\boldsymbol{d}^2, \widetilde{H}] = 2\boldsymbol{d}^2 + 2(\widetilde{X}^{(-)})^2 \\
&[\boldsymbol{d}^2, \boldsymbol{t}^2] = 4(\widetilde{X}^{(+)})^2 && \boldsymbol{d}^2\widetilde{X}^{(\pm)} = \widetilde{X}^{(\pm)}\boldsymbol{t}^2.
\end{aligned} \tag{A1}$$

It is useful to add to the relations (A1) the formula

$$\boldsymbol{d}^{2n} = \boldsymbol{t}^{2n} + 2n\widetilde{X}^{(+)}\boldsymbol{t}^{2(n-1)} \tag{A2}$$

which is easily provable by induction. Indeed, we have

$$\boldsymbol{d}^{2(n+1)} = \boldsymbol{d}^2\boldsymbol{d}^{2n} = (\boldsymbol{t}^2 + 2\widetilde{X}^{(+)})(\boldsymbol{t}^{2n} + 2n\widetilde{X}^{(+)}\boldsymbol{t}^{2(n-1)}) = \boldsymbol{t}^{2(n+1)} + 2(n+1)\widetilde{X}^{(+)}\boldsymbol{t}^{2n}.$$

The commutators (A1) and formula (A2) imply the following useful auxiliary relations. Denote $\boldsymbol{S} = \cosh(h\boldsymbol{t})$, $\boldsymbol{K} = \boldsymbol{t}^{-1}\sinh(h\boldsymbol{t})$. We have

$$\begin{aligned}
&[\widetilde{X}^{(\pm)}, \boldsymbol{S}] = h\widetilde{X}^{(\pm)}\boldsymbol{K}\widetilde{X}^{(+)} \\
&[\boldsymbol{S}, \widetilde{H}] = h\boldsymbol{t}^2\boldsymbol{K} + h\widetilde{X}^{(-)}\boldsymbol{K}\widetilde{X}^{(-)} \\
&[\widetilde{X}^{(\pm)}, \boldsymbol{K}] = \widetilde{X}^{(\pm)}\boldsymbol{t}^{-2}(h\boldsymbol{S} - \boldsymbol{K})\widetilde{X}^{(+)} \\
&[\boldsymbol{K}, \widetilde{H}] = h\boldsymbol{S} - \boldsymbol{K} + \widetilde{X}^{(-)}\boldsymbol{t}^{-2}(h\boldsymbol{S} - \boldsymbol{K})\widetilde{X}^{(-)}.
\end{aligned} \tag{A3}$$

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